Increasing and Decreasing Functions

Definition

A function f is increasing on an interval if, for all x_1, x_2 in interval,

preserves inequality

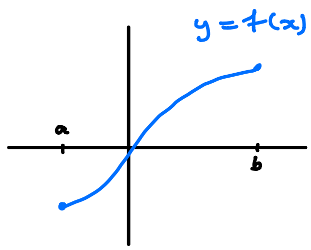
$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

A function f is decreasing on an interval if, for all x_1, x_2 in interval

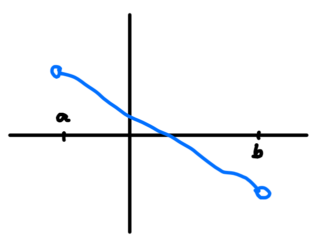
reverses inequality.

$$x_1 < x_2 \implies f(x_1) > f(x_2)$$

Basic Pictures



$\implies$ f increasing on $[a, b]$



$\implies$ f decreasing on (a, b)

Intuition: f is increasing on interval if it has positive slope at each point.

f is decreasing on interval if it has negative slope at each point

Derivative Test for Increasing / Decreasing

Suppose f is differentiable at each point in (a, b). Then

1/ $f'(x) > 0$ for all x in (a,b) $\implies$ f increasing on (a,b)

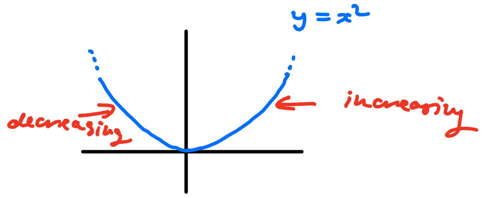
2/ $f'(x) < 0$ for all x in (a,b) $\implies$ f decreasing on (a,b)

3/ $f'(x) = 0$ for all x in (a,b) $\implies$ f constant on (a,b)

Example $f(x) = x^2 \Rightarrow f'(x) = 2x$

$2x < 0$ if x in $(-\infty, 0) \Rightarrow f$ decreasing on $(-\infty, 0)$

$2x > 0$ if x in $(0, \infty) \Rightarrow f$ increasing on $(0, \infty)$



Remark So to determine whether $f(x)$ is increasing / decreasing we need to determine where $f'(x)$ is positive / negative.

Sign Analysis

Let g be a function. Assume that $g(x)$ changes sign at $x = c$. ie

$+ \quad c \quad -$: $g(x) > 0$, $g(x) < 0$ or $- \quad c \quad +$: $g(x) < 0$, $g(x) > 0$

Then either

A/ $g(c) = 0$ or   B/ g discontinuous at $x = c$ (e.g. $g(c)$ undefined)

Example Find where $g(x) = \dfrac{(x+1)(x-2)}{(x-3)(x+2)}$ is positive / negative

A/ $g(x) = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, 2$

B/ g discontinuous $\Rightarrow g$ undefined $\Rightarrow (x-3)(x+2) = 0 \Rightarrow x = -2, 3$

rational function

$+ \quad -2 \quad - \quad -1 \quad + \quad 2 \quad - \quad 3 \quad +$

$g(-3) = \dfrac{-2 \times -5}{-6 \times -1} > 0$ $g(0) = \dfrac{-2}{-6} > 0$ $g(4) = \dfrac{5 \times 2}{1 \times 6} > 0$

$g(-1.5) = \dfrac{-0.5 \times -3.5}{-4.5 \cdot 0.5} < 0$ $g(2.5) = \dfrac{3.5 \times 0.5}{-0.5 \times 4.5} < 0$

Strategy to Find where f Increasing / Decreasing

1/ Calculate $f'(x)$, and determine all c such that
A/ $f'(c) = 0$ or B/ f' discontinuous at c (e.g. $f'(c)$ DNE)

2/ Using these points do sign analysis of f' using test values.

3/ Relate this back to f.

Remark f can change from increasing to decreasing, or vice versa, only at the points found in 1/.

Example

Determine where $f(x) = \dfrac{x^2+12}{x+2}$ is increasing / decreasing.

1/

$$f(x) = \frac{u(x)}{v(x)}, \quad u(x) = x^2 - 1, \quad v(x) = x+2$$

$$\Rightarrow \quad u'(x) = 2x, \quad v'(x) = 1$$

$$\Rightarrow \quad f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x))^2} = \frac{2x(x+2) - (x^2+12)\cdot 1}{(x+2)^2}$$

$$= \frac{x^2+4x-12}{(x+2)^2}$$

A/ $f'(x) = 0 \iff x^2+4x-12 = 0 \iff (x+6)(x-2) = 0$
$\iff x = 2, -6$

f' rational

B/ f' discontinuous $\iff$ f' undefined $\iff (x+2)^2 = 0 \iff x = -2$

2/

$$+ \quad -6 \quad - \quad -2 \quad - \quad 2 \quad + \qquad f'(x)$$

$f'(-3) = \frac{3 \times (-5)}{(-1)^2} < 0$ $\qquad f'(3) = \frac{9 \times 1}{5^2} > 0$

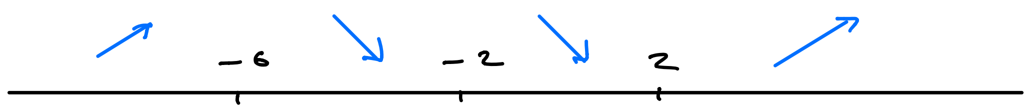
$f'(-7) = \frac{(-1) \times (-9)}{(-5)^2} > 0$ $\qquad f'(0) = -\frac{12}{2^2} < 0$

3/ $f'(x) > 0$ on $(-\infty, -6) \Rightarrow f(x)$ increasing on $(-\infty, -6)$

$f'(x) < 0$ on $(-6, -2) \Rightarrow f(x)$ decreasing on $(-6, -2)$

$f'(x) < 0$ on $(-2, 2) \Rightarrow f(x)$ decreasing on $(-2, 2)$

$f'(x) > 0$ on $(2, \infty) \Rightarrow f(x)$ increasing on $(2, \infty)$



If A/ $f'(c) = 0$ or B/ f' discontinuous at $x = c$
<u>and</u> <u>$f(c)$ exists</u> we say c is a <u>critical number</u> of f.

In the above example $-6, 2$ are critical numbers, whereas -2 is <u>not</u>

<u>Example</u>

$f(x) = (x-1)^{2/3}$. $\Rightarrow$ Domain is all real numbers

$$f'(x) = \tfrac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

$f'(x) = 0 \Rightarrow \frac{2}{3(x-1)^{1/3}} = 0$ has no solutions.

$f'(x)$ discontinuous when $3(x-1)^{1/3} = 0 \Rightarrow x = 1$

$\Rightarrow x = 1$ only critical number

C.N.

$f'(x)$

$-$ 1 $+$

$f'(0) = \dfrac{2}{3(-1)^{1/3}} = \dfrac{2}{-3} < 0$ $f'(2) = \dfrac{2}{3(2-1)^{1/3}} = \dfrac{2}{3} > 0$

$\Rightarrow f$ decreasing on $(-\infty, 1)$

f increasing on $(1, \infty)$.